

Simple portfolio optimization with Mbasic

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The problem is to minimize the variance of the yield of a portfolio composed of n assets, given a desired expected yield. Having solved the problem, a table of the minimal variance (or standard deviation for better understanding) for different expected yields gives a basis for choosing the composition of the portfolio. I.e. it illustrates the trade-off between yield and risk.

Each asset unit is worth 1 dollar. The value after one year for asset i is X_i with $P(X_i \geq 0) = 1$. Set

$$\mu_i = E[X_i] > 0, \quad (i = 1, 2, \dots, n)$$

$$v_{rk} = \text{Cov}(X_r, X_k)$$

Put c_i dollars in asset i , with $c_i \geq 0$.

Search $c_1, c_2, \dots, c_n$ so that $\text{Var} \left[\sum_{i=1}^n c_i X_i \right]$ is minimal, under the restrictions

$$\sum_{i=1}^n c_i = 1 \quad (1 \text{ dollar available in total})$$

$$E \left[\sum_{i=1}^n c_i X_i \right] = \sum_{i=1}^n c_i \mu_i = E_1$$

We must have $\min\{\mu_1, \dots, \mu_n\} \leq E_1 \leq \max\{\mu_1, \dots, \mu_n\}$.

Let

$$\mathbf{e} = [1 \quad 1 \quad \dots \quad 1]^T$$

$$\boldsymbol{\mu} = [\mu_1 \quad \mu_2 \quad \dots \quad \mu_n]^T$$

$$\mathbf{V} = \{v_{rk}\}$$

$$\mathbf{c} = [c_1 \quad c_2 \quad \dots \quad c_n]^T$$

We have

$$\text{Var} \left[\sum_{i=1}^n c_i X_i \right] = \mathbf{c}^T \mathbf{V} \mathbf{c} = \sum_{r=1}^n \sum_{k=1}^n c_r c_k v_{rk}$$

Using Lagrange multipliers we set

$$F(\mathbf{c}) = \mathbf{c}^T \mathbf{V} \mathbf{c} + \lambda_1(1 - c_1 - c_2 - \dots - c_n) + \lambda_2(E_1 - c_1\mu_1 - c_2\mu_2 - \dots - c_n\mu_n)$$

Solve

$$\frac{\partial F(\mathbf{c})}{\partial c_i} = \sum_{r=1}^n \sum_{k=1}^n v_{rk} \frac{\partial (c_r c_k)}{\partial c_i} - \lambda_1 - \lambda_2 \mu_i = 0, \quad i = 1, 2, \dots, n$$

under the restrictions. The condition $c_i \geq 0$ for all i is not included in the restrictions, since this is a simple application. Instead the sample Mbasic program handles possible $c_i < 0$ by optimizing on a subset of the i 's.

We obtain

$$\frac{\partial F(\mathbf{c})}{\partial c_i} = 2 \sum_{r=1}^n v_{ri} c_r - \lambda_1 - \lambda_2 \mu_i$$

Put

$$\alpha_1 = \lambda_1/2$$

$$\alpha_2 = \lambda_2/2$$

$$\mathbf{b} = \begin{bmatrix} \alpha_1 + \alpha_2\mu_1 \\ \alpha_1 + \alpha_2\mu_2 \\ \dots \\ \alpha_1 + \alpha_2\mu_{n-1} \\ \alpha_1 + \alpha_2\mu_n \end{bmatrix}$$

Thus, we have

$$\mathbf{b} = \alpha_1 \mathbf{e} + \alpha_2 \mathbf{\mu}$$

Solve

$$\mathbf{V}\mathbf{c} = \mathbf{b}$$

$$\mathbf{e}^T \mathbf{c} = 1$$

$$\mathbf{\mu}^T \mathbf{c} = E_1$$

That gives, if $\mathbf{V}$ is nonsingular,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{b}$$

$$\mathbf{e}^T \mathbf{c} = \mathbf{e}^T \mathbf{V}^{-1} \mathbf{b} = 1$$

$$\mathbf{\mu}^T \mathbf{c} = \mathbf{\mu}^T \mathbf{V}^{-1} \mathbf{b} = E_1$$

so that

$$\mathbf{e}^T \mathbf{V}^{-1} [\alpha_1 \mathbf{e} + \alpha_2 \mathbf{\mu}] = 1$$

$$\mathbf{\mu}^T \mathbf{V}^{-1} [\alpha_1 \mathbf{e} + \alpha_2 \mathbf{\mu}] = E_1$$

i.e.

$$\alpha_1 \mathbf{e}^T \mathbf{V}^{-1} \mathbf{e} + \alpha_2 \mathbf{e}^T \mathbf{V}^{-1} \mathbf{\mu} = 1$$

$$\alpha_1 \mathbf{\mu}^T \mathbf{V}^{-1} \mathbf{e} + \alpha_2 \mathbf{\mu}^T \mathbf{V}^{-1} \mathbf{\mu} = E_1$$

where, since $\mathbf{V}^{-1}$ is symmetric,

$$\mathbf{\mu}^T \mathbf{V}^{-1} \mathbf{e} = \mathbf{e}^T \mathbf{V}^{-1} \mathbf{\mu}$$

Set

$$F = \mathbf{e}^T \mathbf{V}^{-1} \mathbf{e}$$

$$G = \mathbf{e}^T \mathbf{V}^{-1} \mathbf{\mu}$$

$$H = \mathbf{\mu}^T \mathbf{V}^{-1} \mathbf{\mu}$$

Then we get the linear equations

$$\alpha_1 F + \alpha_2 G = 1$$

$$\alpha_1 G + \alpha_2 H = E_1$$

with solution

$$\alpha_1 = \frac{GE - H}{G^2 - FH}$$

$$\alpha_2 = \frac{G - EF}{G^2 - FH}$$

and finally the desired solution

$$\mathbf{c} = \mathbf{V}^{-1} [\alpha_1 \mathbf{e} + \alpha_2 \mathbf{\mu}]$$